Optimal strategies for dealing with store discounts

John S. Croucher
Macquarie University, Sydney
<jcrouche@gsm.mq.edu.au>

Introduction

While recently shopping in Adelaide for a pair of shoes, I came across a store that advertised 'second pair half price' with the fine print stating that the 'second pair' was the pair of lesser value. Intrigued by such a fine deal, I enquired about the situation if I were to purchase more than two pairs. The reply was that if I purchased three pairs then the cheapest pair would be half price, four pairs would mean the cheapest two pairs would be half price and so on. In mathematical terms, the deal for a customer purchasing n pairs of shoes was:

Total price = full price of the most expensive $n/2$ pairs plus half the price of the least expensive $n/2$ pairs if n is even
= full price of the most expensive $(n + 1)/2$ pairs plus half the price of the least expensive $(n - 1)/2$ pairs if n is odd.

I also discovered that if more than two pairs were purchased then they could be broken up into multiple transactions of any combination. This naturally led to my wondering what would be the optimal way to mix these transactions for multiple purchases so as to minimise the amount paid. (The kindly assistant informed me that in her view it did not make any difference how the purchases were broken up since it would still cost the same!)

In the following calculations, in the case of purchasing n pairs of shoes, it is assumed that the dearest pair cost x_1, the next dearest pair x_2 and so on. That is:

$$x_1 \geq x_2 \geq x_3 \geq x_4 \geq \ldots\ x_{n-1} \geq x_n$$

Before determining a general strategy, it is instructive to consider the first few cases to determine if a pattern appears. In all cases the option of making n single purchases for n pairs of shoes can be ignored, as it will never be superior since no discount at all will apply

Two pairs (n = 2)

This is the simplest case. The two pairs are purchased in a single transaction with a total price of $x_1 + 0.5x_2$.

Three pairs (n = 3)

According to the store rules, making a single transaction would cost $x_1 + x_2 + 0.5x_3$ since the cheapest pair is half price. There are three ways in which the purchases can be broken into two transactions. These, along with the corresponding total cost, are shown in Table 1.

Table 1. The total cost of various transactions for three pairs of shoes.

Combination	Transaction 1	Transaction 2	Total cost
1	x_1, x_2	x_3	$x_1 + \mathrm{x}_3 + 0.5x_2$
2	x_1, x_3	x_2	$x_1 + x_2 + 0.5x_3$
3	x_2, x_3	x_1	$x_1 + x_2 + 0.5x_3$

From Table 1 it can be seen that combinations 2 and 3 cost the same as purchasing all three pairs in one transaction. Since $x_2 \geq x_3$, it can be easily shown that the cheapest option is transaction 1 with a saving of $0.50(x_2 - x_3)$. In practical terms, this means combining the two most expensive pairs in one transaction and the least expensive in a separate transaction.

Four pairs (n = 4)

According to store rules, purchasing all four pairs in a single transaction would cost $x_1 + x_2 + 0.5x_3 + 0.5x_4$. This time there are seven ways of splitting the transactions as shown in Table 2.

Table 2. The total cost of various transactions for four pairs of shoes.

Combination	Transaction 1	Transaction 2	Total cost
1	x_1, x_2, x_3	x_4	$x_1 + x_2 + x_4 + 0.5x_3$
2	x_1, x_2, x_4	x_3	$x_1 + x_2 + x_3 + 0.5x_4$
3	x_1, x_3, x_4	x_2	$x_1 + x_2 + x_3 + 0.5x_4$
4	x_2, x_3, x_4	x_1	$x_1 + x_2 + x_3 + 0.5x_4$
5	x_1, x_2	x_3, x_4	$x_1 + x_3 + 0.5x_2 + 0.5x_4$
6	x_1, x_3	x_2, x_4	$x_1 + x_2 + 0.5x_3 + 0.5x_4$
7	x_1, x_3	x_1, x_4	$x_1 + x_2 + 0.5x_3 + 0.5x_4$

Since $x_4 \le x_3$, combination 1 is cheaper than either of combinations 2, 3 or 4 (which all cost the same); but combinations 6 and 7 (which both cost the same and is the same cost of purchasing them all in a single transaction) are clearly both cheaper than combination 1. Since $x_3 \le x_2$, the cheapest mixture of all is combination 5. This means that the greatest savings can be made by having one transaction of the two most expensive pairs and the other transaction of the two cheapest pairs. It can easily be shown that this represents a saving of $0.50(x_2 - x_3)$ over simply purchasing all four pairs in one transaction. Note that this expression is the same as for three pairs.

n pairs

By now a pattern is beginning to emerge as to the optimal strategy and its proof is left as an exercise. In words, however, it can be expressed as below:

If n is even, make $n/2$ transactions of the type
(x_1, x_2), (x_3, x_4), (x_5, x_6), (x_7, x_8), … (x_{n-1}, x_n)
The total costs are:

$$\text{Cost of optimal strategy} = \sum_{i=1}^{\frac{n}{2}} x_{2i-1} + 0.5\sum_{i=1}^{\frac{n}{2}} x_{2i} \qquad (1)$$

If the n pairs are bought in a single transaction, then the total cost will be:

$$\text{Cost of a single transaction} = \sum_{i=1}^{\frac{n}{2}} x_i + 0.5 \sum_{i=\frac{n}{2}+1}^{n} x_i \qquad (2)$$

Example

A customer decides to purchase six pairs of shoes with ticket prices \$40, \$28, \$70, \$14, \$85 and \$56. Find:

(a) The cost of purchasing these on a single transaction
(b) The cost of purchasing these with an optimal split of transactions
(c) The percentage savings made by using (b) over (a)

Solution

Ordering the prices (in \$) yields:

$x_1 = 85$, $x_2 = 70$, $x_3 = 56$, $x_4 = 40$, $x_5 = 28$, $x_6 = 14$

(a) From (2), the cost (in \$) of making a single transaction
= 85 + 70 + 56 + 20 + 14 + 7
= 252

(b) From (1), the cost (in $) of making three transactions
= 85 + 35 + 56 + 20 + 28 + 7
= 231

(c) From (a) and (b) the actual savings
= $252 – $231
= $21.
The percentage savings is therefore 100 × ($21/$252) = 8.3%

If n is odd, the optimal strategy is to make $(n+1)/2$ transactions of the type (x_1, x_2), (x_3, x_4), (x_5, x_6)… (x_{n-2}, x_{n-1}), x_n. That is, it is basically the same pattern as when n is even, except that the cheapest pair is made as a separate transaction.

The total costs are shown in (3 and (4).

$$\text{Cost of optimal strategy} = \sum_{i=1}^{\frac{(n+1)}{2}} x_{2i-1} + 0.5 \sum_{i=1}^{\frac{(n-1)}{2}} x_{2i} \qquad (3)$$

If the n pairs are bought in a single transaction, then the total cost will be:

$$\text{Cost of a single transaction} = \sum_{i=1}^{\frac{(n+1)}{2}} x_i + 0.5 \sum_{i=\frac{(n+3)}{2}}^{n} x_i \qquad (4)$$

Remarks

Despite the assurances of store employees, whenever a deal is on offer it always pays to apply a little logic (and mathematics) to see just how to use it most effectively. The use of information to gain an advantage has been demonstrated in a number of similar problems, such as the three box problem in which a game show host gives seemingly useless information to try to dissuade a contestant from selecting a particular box that may or may not contain a valuable prize. Croucher and Byun (1998) shows just how this information can be used to increase the probability of winning. Once again, the mathematics involved was not too difficult but absolutely essential.

Reference

Croucher, J. & Byun, K. (1998). The game show dilemma. *Economics, 34* (2), 7–9